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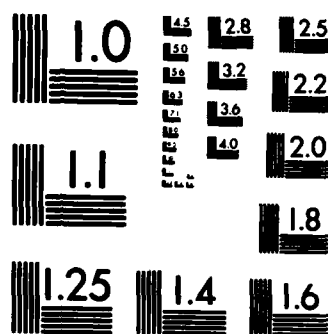
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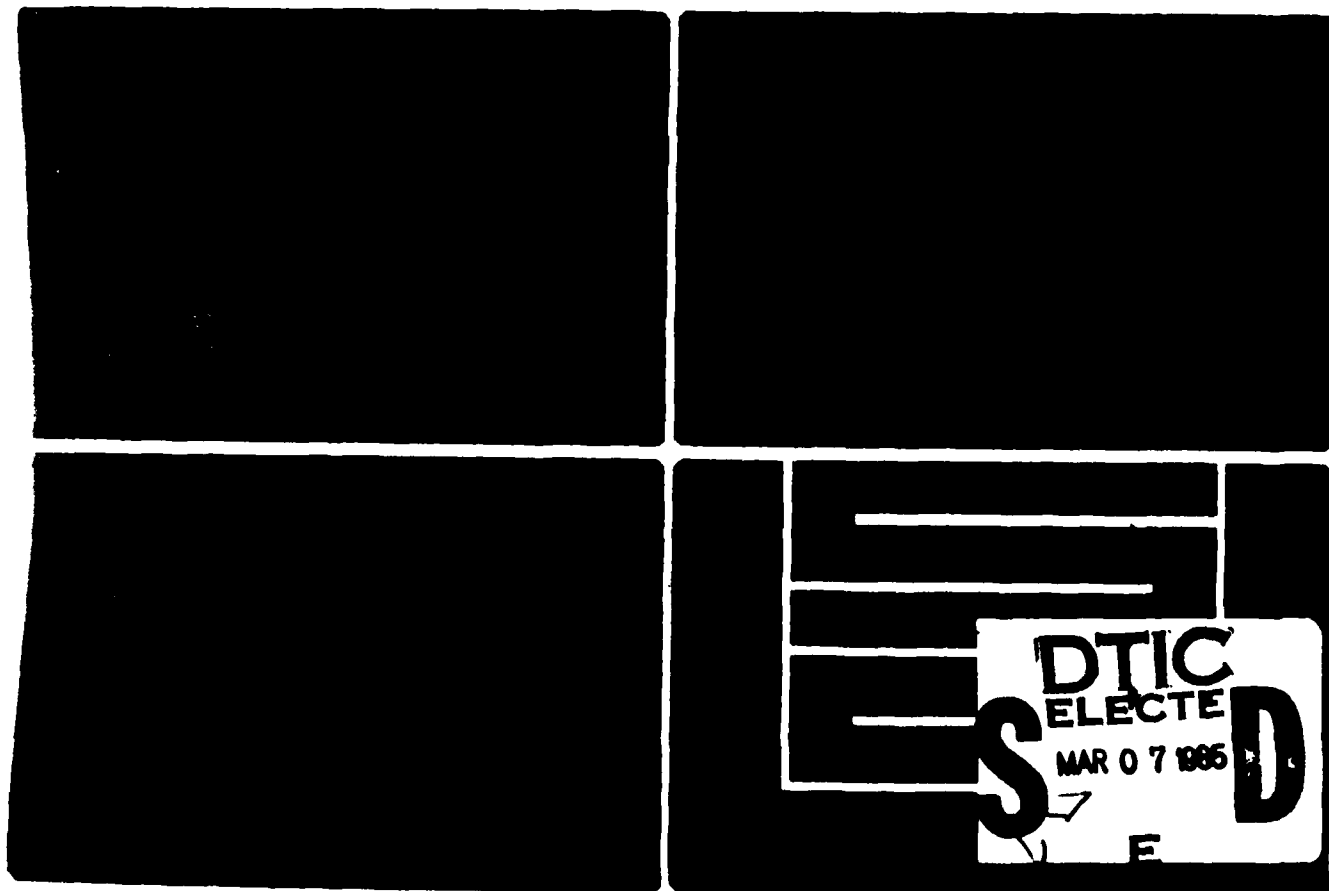
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RATE FUNCTIONS WHEN SAMPLES ARE CENSORED*

by

W. J. Padgett

University of South Carolina
Statistics Technical Report No. 103
62G05-12



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* This work was supported by the U. S. Air Force Office of Scientific
Research and Army Research Office under grant AFOSR-84-0156.

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE					
4. PERFORMING ORGANIZATION REPORT NUMBER(S) Stat. Tech. Rep. No. 103 (62G05-12)			5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 85 - 0107		
6a. NAME OF PERFORMING ORGANIZATION Department of Mathematics and Statistics		6b. OFFICE SYMBOL (If applicable)		7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research	
6c. ADDRESS (City, State and ZIP Code) University of South Carolina Columbia SC 29208				7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR		8b. OFFICE SYMBOL (If applicable) NM		9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-84-0156	
8c. ADDRESS (City, State and ZIP Code) Bolling AFB DC 20332-6448		10. SOURCE OF FUNDING NOS.			
		PROGRAM ELEMENT NO. 61102F		PROJECT NO. 2304	TASK NO. A5
		WORK UNIT NO.			
11. TITLE (Include Security Classification) NONPARAMETRIC ESTIMATION OF DENSITY AND HAZARD RATE FUNCTIONS WHEN SAMPLES ARE CENSORED					
12. PERSONAL AUTHOR(S) W. J. Padgett					
13a. TYPE OF REPORT Technical		13b. TIME COVERED FROM _____ TO _____		14. DATE OF REPORT (Yr., Mo., Day) 1985 January 11	
				15. PAGE COUNT 30	
16. SUPPLEMENTARY NOTATION					
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB GR.	Nonparametric density estimation; Random censorship; Failure rate; Kernel density estimator; Likelihood methods. 		
19. ABSTRACT (Continue on reverse if necessary and identify by block number) Many of the methods of nonparametric density and hazard rate estimation from right-censored observations are discussed. These include histogram and kernel-type procedures, likelihood methods, Fourier series methods, and Bayesian nonparametric approaches. Examples of kernel density estimates are given for mechanical switch life data where data-based choices of the bandwidth values are used. <i>Originator supplied keywords included.</i>					
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS <input type="checkbox"/>			21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a. NAME OF RESPONSIBLE INDIVIDUAL CPT Brian W. Woodruff			22b. TELEPHONE NUMBER (Include Area Code) (202) 767-5027		22c. OFFICE SYMBOL NM

Abstract

Many of the methods of nonparametric density and hazard rate estimation from right-censored observations are discussed. These include histogram and kernel-type procedures, likelihood methods, Fourier series methods, and Bayesian nonparametric approaches. Examples of kernel density estimates are given for mechanical switch life data where data-based choices of the bandwidth values are used.

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1. Introduction

A common and very old problem in statistics is the estimation of an unknown probability density function. In particular, the problem of nonparametric probability density estimation has been studied for many years. Summaries of results on nonparametric density estimation based on complete (uncensored) random samples have been listed recently by several authors, including Fryer [18], Tapia and Thompson [52], Wertz and Schneider [60], and Bean and Tsokos [2]. Also, a review of results for censored samples has been given by Padgett and McNichols [39]. In addition to its importance in theoretical statistics, nonparametric density estimation has been utilized in hazard analysis, life testing, and reliability, as well as in the areas of nonparametric discrimination and high energy physics [20].

→ The purpose of this article is to present the different types of nonparametric density estimates that have been proposed for the situation that the sample data are censored or incomplete. This type of data arises in many life testing situations and is common in survival analysis problems, (see Lagakos [25] and Kalbfleisch and Prentice [21], for example). In many of these situations, some observations may be censored or truncated from the right, referred to as right-censorship. This occurs often in medical trials when the patients may enter treatment at different times and then either die from the disease under investigation or leave the study before its conclusion. A similar situation may occur in industrial life testing when items are removed from the test at random times for various reasons. It is of interest to be able to estimate nonparametrically the unknown density of the lifetime random variable from this type of data without ignoring or discarding the right-censored information. The development of

such nonparametric density estimators has only occurred in the past six or seven years and the avenues of investigation have been similar to those for the complete sample case, except that the problems are generally more difficult mathematically.

The various types of estimators from right-censored samples that have been proposed in the literature will be indicated and briefly discussed here. They include histogram-type estimators, kernel-type estimators, maximum likelihood estimators, Fourier series estimators, and Bayesian estimators. In addition, since the hazard rate function estimation problem is closely related to the density estimation problem, various types of nonparametric hazard rate estimators from right-censored data will be briefly mentioned. Due to their computational simplicity and other properties, the kernel-type density estimators will be emphasized, and some examples will be given in Section 7.

Before beginning the discussion of the various estimators, in the next section the required definitions and notation will be presented.

2. Notation and Preliminaries

Let $X_1^0, X_2^0, \dots, X_n^0$ denote the true survival times of n items or individuals which are censored on the right by a sequence U_1, U_2, \dots, U_n which in general may be either constants or random variables. It is assumed that the X_i^0 's are nonnegative independent identically distributed random variables with common unknown distribution function F^0 . For the problem of density estimation, it is assumed that F^0 is absolutely continuous with density f^0 . The corresponding hazard rate function is defined by $r^0 = f^0/(1-F^0)$.

The observed right-censored data are denoted by the pairs (X_i, Δ_i) , $i=1, \dots, n$, where

$$X_i = \min\{X_i^0, U_i\}, \quad \Delta_i = \begin{cases} 1 & \text{if } X_i^0 \leq U_i \\ 0 & \text{if } X_i^0 > U_i \end{cases}.$$

Thus, it is known which observations are times of failure or death and which ones are censored or loss times. The nature of the censoring mechanism depends on the U_i 's: (i) If U_1, \dots, U_n are fixed constants, the observations are time-truncated. If all U_i 's are equal to the same constant, then the case of Type I censoring results. (ii) If all $U_i = X_{(r)}^0$, the r th order statistic of X_1^0, \dots, X_n^0 , then the situation is that of Type II censoring. (iii) If U_1, \dots, U_n constitute a random sample from a distribution H (which is usually unknown) and are independent of X_1^0, \dots, X_n^0 , then (X_i, Δ_i) , $i=1, 2, \dots, n$, is called a randomly right-censored sample.

The random censorship model (iii) is attractive because of its mathematical convenience. Many of the estimators discussed later are based on this model. Assuming (iii), $\Delta_1, \dots, \Delta_n$ are independent Bernoulli random variables and the distribution function F of each X_i , $i=1, \dots, n$, is given by $1-F = (1-F^0)(1-H)$. Under the Koziol and Green [24] model of random censorship, which is the proportional hazards assumption of Cox [7], it is assumed that there is a positive constant β such that $1-H = (1-F^0)^\beta$. Then by a result of Chen, Hollander, and Langberg [6], the pairs (X_i^0, U_i) , $i=1, \dots, n$, follow the proportional hazards model if and only if (X_1, \dots, X_n) and $(\Delta_1, \dots, \Delta_n)$ are independent. This Koziol-Green model of random censorship arises in several situations (Efron [11], Csörgő and Horváth [8], Chen, Hollander and Langberg [6]). Note that β is a censoring coefficient since $a = P(X_i^0 \leq U_i) = (1+\beta)^{-1}$, which is the probability of an uncensored observation.

Based on the censored sample (X_i, Δ_i) , $i=1, \dots, n$, a popular estimator of the survival probability $S^0(t) = 1-F^0(t)$ at $t \geq 0$ is the product-limit

estimator, proposed by Kaplan and Meier [22] as the "nonparametric maximum likelihood estimator" of S^0 . This estimator was shown to be "self-consistent" by Efron [11]. Let (Z_i, Δ_i') , $i=1, \dots, n$, denote the ordered X_i 's along with their corresponding Δ_i 's. A value of the censored sample will be denoted by the corresponding lower case letters (x_i, δ_i) or (z_i, δ_i') for the unordered or ordered sample, respectively. The product-limit estimator of S^0 is defined by [11]

$$\hat{P}_n(t) = \begin{cases} 1, & 0 \leq t \leq Z_1 \\ \prod_{i=1}^{k-1} \left(\frac{n-i}{n-i+1} \right)^{\Delta_i'}, & t \in (Z_{k-1}, Z_k], k=2, \dots, n. \\ 0, & t > Z_n. \end{cases}$$

Denote the product-limit estimator of $F^0(t)$ by $\hat{F}_n(t) = 1 - \hat{P}_n(t)$, and let s_j denote the jump of \hat{P}_n (or \hat{F}_n) at Z_j , that is,

$$s_j = \begin{cases} 1 - \hat{P}_n(Z_2), & j=1 \\ \hat{P}_n(Z_j) - \hat{P}_n(Z_{j+1}), & j=2, \dots, n-1 \\ \hat{P}_n(Z_n), & j=n. \end{cases} \quad (2.1)$$

Note that $s_j = 0$ if and only if $\Delta_j' = 0$, $j < n$, that is, if Z_j is a censored observation.

The product-limit estimator has played a central role in the analysis of censored survival data (Miller [36]), and its properties have been studied extensively by many authors, for example, Breslow and Crowley [4], Földes, Rejtő and Winter [15], and Wellner [59]. Many of the nonparametric density estimators from right-censored data are naturally based on the product-limit estimator, beginning with the histogram-type and kernel-type estimators.

3. Histogram and Kernel Estimators

One of the simplest nonparametric estimators of the density function for randomly right-censored samples is the histogram estimator. Although they are simple to compute, histogram estimators are not smooth and are generally not suited to sophisticated inference procedures.

Estimation of the density function and hazard rate of survival time based on randomly right-censored data was apparently first studied by Gehan [19]. The life table estimate of the survival function was used to estimate the density f^0 as follows: The observations (x_i, δ_i) , $i=1, \dots, n$, were grouped into k fixed intervals $[t_1, t_2), [t_2, t_3), \dots, [t_k, \infty)$, with the finite widths denoted by $h_i = t_{i+1} - t_i$, $i=1, \dots, k-1$. Letting n'_i denote the number of individuals alive at time t_i , L_i be the number of individuals censored (lost or withdrawn from the study) in the interval $[t_i, t_{i+1})$, and d_i be the number of individuals dying or failing in the i th interval (where time to death or failure is recorded from time of entry into the study), define $\hat{q}_i = d_i/n_i$ and $\hat{p}_i = 1 - \hat{q}_i$, where $n_i = n'_i - L_i/2$. Therefore, \hat{q}_i is an estimate of the probability of dying or failing in the i th interval, given exposure to risk in the i th interval. Let $\hat{\Pi}_i = \hat{p}_{i-1} \hat{\Pi}_{i-1}$, where $\hat{\Pi}_1 \equiv 1$. Gehan's estimate of f^0 at the midpoint t_{mi} of the i th interval is then

$$\hat{f}(t_{mi}) = \frac{\hat{\Pi}_i - \hat{\Pi}_{i+1}}{h_i} = \frac{\hat{\Pi}_i \hat{q}_i}{h_i}, \quad i=1, \dots, k-1.$$

An expression for estimating the large sample approximation to the variance of $\hat{f}(t_{mi})$ was also given in [19].

Using the product-limit estimator \hat{F}_n of F^0 , Földes, Rejtő, and Winter [16] defined a histogram estimator of f^0 on a specified interval $[0, T]$, $T > 0$.

For integer $n > 0$, let $0 = t_0^{(n)} < t_1^{(n)} < \dots < t_{v_n}^{(n)} = T$ be a partition of $[0, T]$ into n subintervals $I_i^{(n)}$, where

$$I_i^{(n)} = \begin{cases} [t_{i-1}^{(n)}, t_i^{(n)}), & 1 \leq i < v_n \\ [t_{v_n-1}^{(n)}, T], & i = v_n. \end{cases}$$

Then their histogram estimator is

$$\hat{f}(x) = \frac{\hat{F}_n(t_i^{(n)}) - \hat{F}_n(t_{i-1}^{(n)})}{t_i^{(n)} - t_{i-1}^{(n)}}, \quad x \in I_i^{(n)}. \quad (3.1)$$

If $x \notin [0, T]$, $\hat{f}(x)$ is either undefined or defined arbitrarily. Notice that if none of the observations are censored, \hat{F}_n reduces to the empirical distribution function, and (3.1) becomes the usual histogram estimator with respect to the given partition. The strong uniform consistency of \hat{f} on $[0, T]$ was proven by Földes, Rejtő, and Winter [16] under some conditions on the partition, provided that f^0 was continuous on $[0, T]$ and $H(T^-) < 1$, where $H(T^-)$ denotes the limit from the left of H at T . This last condition is common in obtaining consistency properties under random right-censorship and insures that uncensored observations can be obtained from the entire interval of interest.

Burke and Horváth [5] defined general density estimators which included histogram-type and kernel-type estimators with appropriate choices of the defining functions. They also obtained asymptotic distribution results for these estimators. In fact, their results were obtained for the more general situation of the k independent competing risks model. When $k=2$, this reduces to the random right-censorship model.

The histogram estimator can be obtained as a special case of the kernel density estimators. The kernel-type estimators have been perhaps the most popular estimators in practice due to their relative computational simplicity, smoothness, and other properties. Kernel-type estimators from randomly right-censored data have been studied only since around 1978, beginning with the work of Blum and Susarla [3]. The investigation of kernel estimators for right-censored samples has been attempted along the same lines as for the complete sample case. However, due to mathematical difficulties introduced by the censoring, some of the analogous theory to the complete sample case has not yet been obtained.

Blum and Susarla [3] generalized the complete sample results of Rosenblatt [45] concerning maximum deviation of density estimates by the kernel method. To define the Blum-Susarla density estimator, let $\{h_n\}$ be a positive sequence, called the bandwidth sequence, such that $\lim_{n \rightarrow \infty} h_n = 0$, and let $N^+(x)$ denote the number of observed X_i 's that are greater than x . Define

$$H_n^*(x) = \prod_{j=1}^n \left\{ \frac{1+N^+(X_j)}{2+N^+(X_j)} \right\} [\Delta_j = 0, X_j \leq x],$$

where $[A]$ denotes the indicator function of the event A . By a modification of the product-limit estimator, it can be shown that H_n^* is a good estimate of $H^* = 1-H$. For a kernel function K satisfying certain conditions, the Blum-Susarla density estimator is given by

$$f_n^*(x) = [n h_n H_n^*(x)]^{-1} \sum_{j=1}^n K\left(\frac{x-X_j}{h_n}\right) [\Delta_j = 1]. \quad (3.2)$$

For example, K can be a bounded density function with support in the interval $[-A, A]$ for some $A > 0$ and absolutely continuous on $[-A, A]$ with derivative

K' which is square integrable on $[-A, A]$. By following standard arguments, $(f^0 H^*)_n(x) \equiv (n h_n)^{-1} \sum_{j=1}^n K((x - X_j)/h_n) [\Delta_j = 1]$ and $H_n^*(x)$ can be shown to be good estimators of $f^0(x) H^*(x)$ and $H^*(x)$, respectively.

This motivates the use of (3.1) as an estimator of $f^0(x)$.

Blum and Susarla also obtain limit theorems for the maximum over a finite interval of a normalized deviation of the density estimator (3.2).

These results are useful for goodness-of-fit tests and tests of hypotheses about the unknown lifetime density f^0 .

It was conjectured by Blum and Susarla [3] that the kernel-type estimator

$$\hat{f}_n(x) = h_n^{-1} \int_{-\infty}^{\infty} K((x-t)/h_n) d F_n^*(t)$$

behaved in the same way as f_n^* , where F_n^* was an estimator of F^0 such as the product-limit estimator. In fact, Földes, Rejtő, and Winter [16] proved uniform almost sure convergence of \hat{f}_n to f^0 when F_n^* was taken to be \hat{F}_n . Specifically, one of their results was that $\sup_{a < x < b} |\hat{f}_n(x) - f^0(x)| \rightarrow 0$ almost surely as $n \rightarrow \infty$ provided f^0 was bounded and had a bounded derivative on (a, b) , $-\infty \leq a < b \leq \infty$, K was right-continuous and of bounded variation, $h_n(n/\log n)^{1/8} \rightarrow \infty$, and $H(T_{F^0}^-) < 1$, where $T_{F^0}^- = \sup\{x: F^0(x) < 1\}$. Again, the last condition insured that observed lifetimes in the entire support of F^0 would be available. It should be noted that if no censoring is present, then

$$\hat{f}_n(x) = h_n^{-1} \int_{-\infty}^{\infty} K((x-t)/h_n) d \hat{F}_n(t) \quad (3.3)$$

reduces to the Parzen [43] estimator.

McNichols and Padgett [32] wrote (3.3) in the form

$$\hat{f}_n(x) = h_n^{-1} \sum_{j=1}^n s_j K[(x - Z_j)/h_n], \quad (3.4)$$

where s_j is given by (2.1). They considered the mean, variance, and mean squared error of (3.4) under the Koziol-Green model of random censorship described in Section 2. This model allowed the expected value of $\hat{f}_n(x)$ to be evaluated by using the independence of (X_1, \dots, X_n) and $(\Delta_1, \dots, \Delta_n)$. In particular, if K is a Borel function such that $\sup |K(t)| < \infty$, $\int_{-\infty}^{\infty} |K(t)| dt < \infty$, $\lim_{t \rightarrow \infty} |t K(t)| = 0$, and $\int_{-\infty}^{\infty} K(t) dt = 1$, then

$$E[\hat{f}_n(x)] = a h_n^{-1} \int_0^{\infty} g_n(t) f(t) K((x-t)/h_n) dt \\ + (1-a) p_n(a) h_n^{-1} E[K((x-Z_n)/h_n)], \quad (3.5)$$

where $a = (1+\beta)^{-1}$, $b = 1-a$, $p_n(a) = \prod_{i=1}^{n-1} [(n-i+b)/(n-i+1)]$,

$$g_n(t) = \sum_{j=1}^n \binom{n-1+b}{j-1} [1-F(t)]^{n-j} [F(t)]^{j-1},$$

$$\binom{n+b}{k} = (n+b)(n+b-1) \dots (n+b-k+1)/k!,$$

$F = 1 - (1-H)(1-F^0)$, and f is a density for F . Furthermore, it was shown that if $h_n \rightarrow 0$, then $\lim_{n \rightarrow \infty} E[\hat{f}_n(x)] = f^0(x)$, $x > 0$. Thus, under the Koziol-Green model, $\hat{f}_n(x)$ is asymptotically unbiased for $f^0(x)$ similar to the complete sample case (the conditions on K and h_n are those imposed by Parzen [43]). Second moment convergence was also obtained under the conditions that $n h_n \rightarrow \infty$ and $b = P(\text{a censored observation}) < 1$ in addition to the conditions required for asymptotic unbiasedness above [32].

For the kernel estimator (3.4), it is desirable to allow the data to play a role in how much smoothing is done. Since, for a fixed n , h_n is the "smoothing constant," it would be reasonable to allow h_n to be a function of the right-censored sample. McNichols and Padgett [35] consider this type of

modification, which extends the work of Wagner [54] to censored data. This modified kernel estimator is

$$f_n(x) = \Gamma_n^{-1} \sum_{j=1}^n s_j K[(x-Z_j)/\Gamma_n], \quad (3.6)$$

where $\Gamma_n \equiv \Gamma_n(X_1, \dots, X_n)$ is some function of the censored data. For this estimator it was shown that if $H(T_{FO}^-) < 1$, K has bounded variation,

$\lim_{|x| \rightarrow \infty} xK(x) = 0$, $\Gamma_n \rightarrow 0$ in probability (almost surely), and $n^{1/2}(\log \log n)^{-1/2} \Gamma_n \rightarrow \infty$ in probability (almost surely), then $f_n(x) \rightarrow f^0(x)$ in probability (almost surely) at each x for which f^0 is continuous.

One choice of Γ_n satisfying the above conditions is as follows:

If $\gamma_n = [n^\alpha]$, $1/2 < \alpha < 1$, where $[\cdot]$ denotes the greatest integer function, let D_{jn} be the distance from Z_j to its γ_n -nearest neighbor among $Z_1, \dots, Z_{j-1}, Z_{j+1}, \dots, Z_n$, $1 \leq j \leq n$, and select Γ_n to be D_{jn} with probability s_j .

The practical choice of the bandwidth h_n for a given censored sample is a problem which must be addressed in order to calculate the kernel estimator. For complete samples, several "data-based" procedures for selecting a "good" value of h_n for a given set of data have been proposed (see Scott and Factor [46], for example). Among these procedures when samples are right-censored, the maximum likelihood approach seems to be feasible. This will be discussed further in Section 6.

With the exception of the expressions for the mean, $E[\hat{f}_n(x)]$, in (3.5) and for $E[\hat{f}_n^2(x)]$ under the Koziol-Green model [32], very little has been done concerning the small-sample properties of \hat{f}_n or any of the other kernel-type density estimators in the censored data case. Padgett and McNichols [40]

have performed Monte Carlo simulations for several parametric families of lifetime distributions, uniform and exponential censoring distributions, several kernel functions, and several bandwidths to determine the small-sample behavior of \hat{f}_n with respect to bias and mean squared error.

For estimating the hazard rate function r^0 from randomly right-censored data, Földes, Rejtő, and Winter [16] considered estimators of the form

$$r_n(x) = \frac{\hat{f}_n(x)}{1 - \hat{F}_n(x) + \frac{1}{n}}, \quad x \geq 0,$$

where \hat{f} denoted either their histogram estimator (3.1) or their kernel-type estimator (3.3). The $1/n$ in the denominator simply prevents dividing by zero. Strong consistency results for r_n similar to those for (3.1) and (3.3) were proven.

McNichols and Padgett [34] considered the kernel-type estimator of r^0 given by

$$r_n(x) = h_n^{-1} \int K((x-t)/h_n) [1 - \hat{F}_n(t)]^{-1} d\hat{F}_n(t),$$

$$x \geq 0 \text{ such that } F(x) < 1,$$

under the Koziol-Green model of random censorship. Expressions for $E[r_n(x)]$ and $\text{var}[r_n(x)]$ were obtained, and it was shown that $r_n(x)$ was asymptotically unbiased, and converged in mean square and in probability to $r^0(x)$, extending Watson and Leadbetter's [55,56] results.

Tanner and Wong [50] also studied a kernel-type estimator of r^0 based on the ordered censored sample (Z_i, Δ_i') , $i=1, \dots, n$, given by

$$\hat{r}(x) = \sum_{j=1}^n (n-j+1)^{-1} \Delta_j' K_{h_n}(x-Z_j),$$

$x \geq 0$ such that $F(x) < 1$,

where K was a symmetric integrable kernel with $K_h(y) = K(y/h)$. They derived expressions for $E[\hat{r}(x)]$ and $\text{var}[\hat{r}(x)]$ and proved under the conditions on K stated by Watson and Leadbetter [55,56] that $\hat{r}(x)$ was asymptotically unbiased if $h_n \rightarrow 0$ and $nh_n \rightarrow \infty$. The conditions assumed here were essentially the same as those required by McNichols and Padgett [34], except for the proportional hazards (Koziol-Green) model assumption which gave somewhat different expressions for the mean and variance. The asymptotic variance was also obtained, and Hajek's projection method was used to establish asymptotic normality under conditions on K , F^0 , H , and h_n . Tanner and Wong [51] studied a class of estimators of the same general form as $\hat{r}(x)$ with K_h replaced by $K_{\underline{\theta}}$, where $\underline{\theta}$ was a positive-valued "smoothing vector" chosen to maximize a likelihood function. Hence, for this estimator the smoothing parameters were chosen based on the observed data.

Tanner [49] considered a modified kernel-type estimator of r^0 in the form

$$\tilde{r}_n(x) = (2R_k)^{-1} \sum_{i=1}^n \frac{\Delta_i'}{n-i+1} K((x-Z_i)/2R_k),$$

where R_k was the distance from x to the k th nearest of the uncensored observations among X_1, \dots, X_n . This estimator allowed the data to play a role in determining the degree of smoothing that would occur in the estimate. Assuming that S^0 and f^0 were continuous in a neighborhood about $x, k = [n^\alpha]$, $\frac{1}{2} < \alpha < 1$, where $[\cdot]$ was the greatest integer function, that K had bounded variation and compact support on the interval $[-1,1]$, and that

r^0 was continuous at x , it was shown that $\tilde{r}_n(x)$ was strongly consistent.

Blum and Susarla [3] considered the estimator (in the notation of Equation (3.2))

$$\hat{r}_n(x) = \frac{(f^0 H^*)_n(x)}{S_n^*(x)}, \quad x \geq 0,$$

where $S_n^*(x) = (\text{number of } Z_j\text{'s} > x)/n$. This estimator was also of the kernel type, and limiting results similar to those stated for the density estimator (3.2) were obtained for \hat{r}_n .

Ramlau-Hansen [44] used martingale techniques to treat the general multiplicative intensity model. His results are very general and include the kernel estimators of hazard rate functions of Földes, Rejtő, and Winter [16] and Yandell [61]. The martingale techniques yielded local asymptotic properties of many of the hazard rate estimators in a simpler manner than classical procedures.

Finally, in a recent paper Liu and Van Ryzin [26] obtained a histogram estimator of the hazard rate function from randomly right-censored data based on spacings in the order statistics. They showed the estimator to be uniformly consistent in a bounded interval and asymptotically normal under suitable conditions. An efficiency comparison of their estimator with the kernel estimator of hazard rate was also given. Also, Liu and Van Ryzin [27] gave the large sample theory for the normalized maximal deviation of a hazard rate estimator under random censoring which was based on a histogram estimate of the subsurvival density of the uncensored observations.

4. Likelihood Methods

One approach to estimating a density function nonparametrically is that of maximum likelihood. Nonparametric maximum likelihood estimates of a

probability density function do not exist in general. That is, the likelihood function for a complete sample is unbounded over the class of all possible densities. However, by suitably restricting the class of densities, a nonparametric maximum likelihood estimator (MLE) may be found within the restricted class. For complete samples, the maximum likelihood estimator of a density g was given by Barlow, Bartholomew, Bremner and Brunk [1] if g was assumed to be either decreasing (nonincreasing) or unimodal with known mode. Wegman [57,58] assumed unimodality with unknown mode and found the MLE of the density and studied its properties for complete samples.

McNichols and Padgett [33] studied maximum likelihood estimation of decreasing or unimodal densities based on arbitrarily right-censored data. The censoring variables U_1, \dots, U_n could be either constants or continuous random variables. They first assumed that f^0 was decreasing (nonincreasing) on $[0, \infty)$ and let F_D be the set of distributions with decreasing left-continuous densities on $[0, \infty)$. For the ordered censored observations (z_i, δ'_i) , $i=1, \dots, n$, the likelihood function was written as

$$L(f^0) = \prod_{i=1}^n [f^0(z_i)]^{\delta'_i} [S^0(z_i)]^{1-\delta'_i},$$

where $S^0 = 1 - F^0$. It was shown that a maximum likelihood estimator of f^0 must be a step function.

The estimator was found by maximizing the likelihood function $L(f^0)$ over F_D subject to the decreasing density constraint. Equivalently, the constrained optimization problem to be solved was

$$\text{maximize}_{y_1, \dots, y_n} \sum_{i=1}^n \{ \delta'_i \log y_i + (1-\delta'_i) \log [1 - \sum_{j=1}^i y_j (z_j - z_{j-1})] \}$$

subject to (i) $y_1 \geq y_2 \geq \dots \geq y_n \geq 0$

$$(ii) \sum_{j=1}^n y_j (z_j - z_{j-1}) \leq 1,$$

where $z_0 \equiv 0$. This function to be maximized was shown to be concave and the problem was shown to have a unique solution, say y_1^*, \dots, y_n^* . Then any density of the form

$$f^*(x) = \begin{cases} 0, & x \leq 0 \\ y_j^*, & z_{j-1} < x \leq z_j, j=1, \dots, n+1 \\ 0, & x > z_{n+1} \end{cases}$$

was a maximum likelihood estimator of f^0 , where y_{n+1}^* , some value less than or equal to y_n^* , and $z_{n+1} (> z_n)$ were chosen so that

$$1 - \sum_{j=1}^n y_j^* (z_j - z_{j-1}) = y_{n+1}^* (z_{n+1} - z_n).$$

Similarly, f^0 was estimated by maximum likelihood assuming that f^0 was increasing (nondecreasing) on $[0, M]$, $M > 0$ known. Then, if M denoted the known mode of the unknown unimodal density, the two maximum likelihood estimators on $[0, M]$ and on (M, ∞) found as above could be combined to estimate the unimodal density. If f^0 was assumed to be unimodal with unknown mode M , then McNichols and Padgett [33] applied the above procedure for known mode, assuming $z_{j-1} < M < z_j$ for each $j=1, \dots, n$, obtaining n solutions for f^0 . These n solutions gave n corresponding values of the likelihood function. The maximum likelihood estimator of f^0 was then taken to be the solution with the largest of the n likelihood values, analogous to Wegman's [57,58] procedure for complete samples.

Another approach to the problem of nonparametric maximum likelihood estimation of a density from complete samples was proposed by Good and Gaskins [20]. This method allowed any smooth integrable function on the interval of interest (a,b) (which may be finite or infinite) as a possible estimator, but added a "penalty function" to the likelihood. The penalty function penalized a density for its lack of smoothness, so that a very "rough" density would have a smaller likelihood than a "smooth" density, and hence, would not be admissible. De Montricher, Tapia, and Thompson [9] proved the existence and uniqueness of the maximum penalized likelihood estimator (MPLE) for complete samples. Lubecke and Padgett [30] assumed that the sample was arbitrarily right-censored, $(X_i, \Delta_i), i=1, \dots, n$, and showed the existence and uniqueness of a solution to the problem:

$$\begin{aligned} &\text{maximize } L(g) \text{ subject to} \\ &g(t) \geq 0 \text{ for all } t \in \Omega, \int_{\Omega} g(t) dt = 1, \\ &\text{and } g \in H(\Omega), \end{aligned} \quad (4.1)$$

where $L(g) = \prod_{i=1}^n [g(x_i)]^{\delta_i} [1-G(x_i)]^{1-\delta_i} \exp[-\phi(g)]$, Ω is a finite or infinite interval, $H(\Omega)$ is a manifold, and G is the distribution function for density g . In particular, letting $u = g^{\frac{1}{2}}$ and using Good and Gaskins' [20] first penalty function, the problem (4.1) becomes:

$$\begin{aligned} \text{maximize } \hat{L}(u) = &\prod_{i=1}^n [u(x_i)]^{\delta_i} [1 - \int_{-\infty}^{x_i} u^2(t) dt]^{\frac{1}{2}(1-\delta_i)} \\ &\times \exp[-2\alpha \int_0^{\infty} (u'(t))^2 dt], \end{aligned} \quad (4.2)$$

where $x_i > 0, i=1, \dots, n, \int_0^{\infty} u^2(t) dt = 1$, and $u(t) \geq 0, t > 0$.

Let $x_{-1} = x_1$ and $\delta_{-1} = \delta_1, i=1, \dots, n$, and define $\bar{u}(x) = u(|x|)$ for

$x \in \mathbb{R} \setminus \{0\}$ and $\bar{u}(0) = \lim_{x \rightarrow 0^+} u(x)$. Then define the following problem:

$$\begin{aligned} \text{maximize } L(\bar{u}) = & \prod_{|i|=1}^n [\bar{u}(x_i)]^{\delta_i} [2 - \int_{-\infty}^{x_i} \bar{u}^2(t) dt]^{\frac{1}{2}(1-\delta_i)} \\ & \times \exp[-2\alpha \int_{-\infty}^{\infty} (\bar{u}'(t))^2 dt], \end{aligned} \quad (4.3)$$

where $\int_{-\infty}^{\infty} \bar{u}^2(t) dt = 2$, $\bar{u} \in H_S \equiv \{g \in H^1(-\infty, \infty) : g(x) = g(-x)\}$, and $H^1(-\infty, \infty)$ is the Sobolev space of real-valued functions such that the function and its first derivative are square integrable.

If u^* solves (4.3), then it can be shown that $u_+^*(t) = u^*(t)$, $t \geq 0$, and $u_+^*(t) = 0$, $t < 0$, solves (4.2). Lubecke and Padgett [30] showed that a solution to (4.3) was a function \bar{u}^* which solves the linear integral equation

$$\begin{aligned} \bar{u}_\lambda(t) = & C(t; \underline{x}, \alpha, \lambda) + (8\alpha\lambda)^{-\frac{1}{2}} \int_0^t \left[\sum_{|i|=1}^n \frac{(1-\delta_i)}{U_{2\lambda}(x_i)} I_{(-\infty, x_i]}(|\tau|) \right] \\ & \times \sinh[(\lambda/2\alpha)^{\frac{1}{2}}(t-\tau)] \bar{u}_\lambda(\tau) d\tau, \end{aligned} \quad (4.4)$$

where the forcing function is defined by

$$\begin{aligned} C(t; \underline{x}, \alpha, \lambda) \equiv & \frac{1}{2} \left\{ \sum_{|i|=1}^n \frac{\delta_i (2\alpha\lambda)^{-\frac{1}{2}}}{\bar{u}_\lambda(x_i)} [\exp(-(\lambda/2\alpha)^{\frac{1}{2}} |t-x_i|) \right. \\ & \left. + \exp(-(\lambda/2\alpha)^{\frac{1}{2}} |t+x_i|)] \right. \\ & \left. - \sum_{|i|=1}^n \frac{(1-\delta_i)}{U_{2\lambda}(x_i)} [\exp(-(\lambda/2\alpha)t) + \exp((\lambda/2\alpha)^{\frac{1}{2}} t)] \right\}, \end{aligned}$$

for a $\lambda > 0$. The integral equation (4.4) can be transformed to a second-order differential equation whose solution \bar{u}^* can be numerically obtained. Then $(\bar{u}_+^*)^2$ is the MPLE of the density f^0 based on the first penalty function of Good and Gaskins.

The nonparametric maximum likelihood estimation of the hazard rate function r^0 based on the arbitrarily right-censored sample (X_i, Δ_i) , $i=1,2,\dots,n$, was considered by Padgett and Wei [41] in the class of increasing failure rate (IFR) distributions. The techniques of order restricted inference were used to obtain the estimator following an argument similar to that of Marshall and Proschan [31] for the complete sample case. A closed form solution to the likelihood function of r^0 subject to the IFR condition was found to be a nondecreasing step function. Small sample properties of their estimator were indicated by a Monte Carlo study. Mykytyn and Santner [37] considered the same problem of maximum likelihood estimation of r^0 under arbitrary right censorship assuming either IFR, decreasing failure rate (DFR), or U-shaped failure rate. Their estimator was essentially equivalent to Padgett and Wei's estimator and was shown to be consistent by using a total time on test transform. This estimator was maximum likelihood in the Kiefer-Wolfowitz sense.

Friedman [17] also considered maximum likelihood estimation from survival data. Let n survival times be observed over a time period divided into $I(n)$ intervals and assume that the hazard rate function of the time to failure of individual j , $r_j(t)$, is constant and equal to $r_{ij} > 0$ on the i th interval. The maximum likelihood estimate $\hat{\lambda}$ of the vector $\lambda = \{\log r_{ij} : j=1,\dots,n; i=1,\dots,I(n)\}$ gave a simultaneous estimate of the hazard rate function. Friedman gave conditions for the existence of $\hat{\lambda}$ and studied the asymptotic properties of linear functionals of $\hat{\lambda}$ in the general case when the true hazard rate is not a step function. This piecewise smooth estimate of the hazard rate can be regarded as giving piecewise smooth density estimates.

5. Some Other Methods

Nonparametric density estimators based on Fourier series representations have been proposed for censored data. Kimura [23] considered the problem of estimating density functions and cumulatives by using estimated Fourier series. A method for generating a useful class of orthonormal families was first developed for the complete sample case and the results were then generalized to the case of censored data. Variance expressions for the quantity $-\int_{-\infty}^{\infty} \phi(x) d\hat{P}_n(x)$ were obtained, where ϕ was chosen so that the variance existed and \hat{P}_n was the product-limit estimator. Finally, Monte Carlo simulation was used to test the methods developed.

Tarter [53] obtained a new maximum likelihood estimator of the survival function S^0 by using Fourier series estimators of the probability densities of the uncensored observations and censored observations separately. That is, the density estimates were \hat{f} and \tilde{f} , obtained from the n_1 observed uncensored X_1 's and the n_2 observed censored X_1 's, respectively, where $n_1 + n_2 = n$. It was shown that as $n \rightarrow \infty$ the new likelihood estimator approached the product-limit estimator from above. It should be noted that the series-type density estimators \hat{f} and \tilde{f} used here were obtained by the usual complete-sample formulas.

The final series-type estimator to be mentioned here is the general estimator of the density in the k competing risks model of Burke and Horváth [5]. It could be considered as a Fourier-type estimator by appropriate choices of the form of the defining functions.

Another method that has been used for estimating hazard rate and density functions is that of Bayesian nonparametric estimation. Since the work of Ferguson [12,13], many authors have been concerned with the Bayesian nonparametric

estimation of a distribution function or related functions with respect to the Dirichlet process or other random probability measures as prior distributions. For censored data Susarla and Van Ryzin [47,48] considered the estimation of the survival function with respect to Dirichlet process priors, while Ferguson and Phadia [14] used neutral to the right processes as prior distributions.

Padgett and Wei [42] obtained Bayesian nonparametric estimators of the survival function, density function, and hazard rate function of the lifetime distribution using pure jump processes as prior distributions on the hazard rate function, assuming an increasing hazard rate. Both complete and right-censored samples were considered. The pure jump process prior was appealing because it had an intuitive physical interpretation as shocks occurring randomly in time that caused the hazard rate to increase a constant small amount at each shock, which also closely approximated the (random) increasing failure rate by a (random) step function.

Dykstra and Laud [10] also considered a prior distribution on the hazard rate function in order to produce smooth nonparametric Bayes estimators. Their prior was an extended gamma process and the posterior distribution was found for right-censored data. The Bayes estimators of the survival and hazard rate functions with respect to a squared error loss were obtained in terms of a one-dimensional integral.

Lo [28,29] estimated densities and hazard rates, as well as other general rate functions, from a Bayesian nonparametric approach by constructing a prior random density as a convolution of a kernel function with the Dirichlet random probability. His estimator of the density with respect to squared error loss was essentially a mixture of an initial or prior guess at the density and a sample probability density function. His technique can be used for complete or censored samples.

6. Numerical Examples of Some Kernel Density Estimators

Of the many types of nonparametric density estimators available, probably the most often used in practice are the kernel-type estimators. They are relatively simple to calculate and can produce smooth, pleasing results. In this section numerical examples will be given for the kernel estimator (3.4) and the modified estimator (3.6) with the nearest neighbor-type procedure for selecting Γ_n .

One problem in using kernel density estimators is that of how to choose the "best" value of the bandwidth h_n to use with a given set of data. This question has been addressed in the complete sample case by several authors (see Scott and Factor [46], for example), and "data-based" choices of h_n have been proposed using maximum likelihood, mean squared error, or other criteria. For the estimator (3.4) no expressions for the mean squared error for finite sample sizes exist at present, except for those very complicated ones given by McNichols and Padgett [32] under the Koziol-Green model. Hence, selection of h_n to minimize mean squared error does not seem to be feasible. However, Monte Carlo simulation results of Padgett and McNichols [40] indicate that at each x there is a value of h_n which minimizes the estimated mean squared error of $\hat{f}_n(x)$ in (3.4). Similar results were also obtained in [40] for the Blum-Susarla estimator $f_n^*(x)$ defined by (3.2). These simulation results indicated a range of values of h_n which gave small estimated mean squared errors of $\hat{f}_n(x)$ and $f_n^*(x)$ at fixed x . The maximum likelihood criterion for selecting h_n for a given censored sample is feasible for \hat{f}_n but does not seem to be tractable, even using numerical methods, for f_n^* due to the complications introduced by the term $H_n^*(x)$ in the likelihood expression. The maximum likelihood

approach will be used in the following example for \hat{f}_n .

Following a similar approach to expressions (2.8) and (2.9) of Scott and Factor [46], consider choosing h_n to be a value of $h \geq 0$ which maximizes the likelihood

$$L(h) = \prod_{i=1}^n [\hat{f}_n(z_i)]^{\delta'_i} [\int_{z_i}^{\infty} \hat{f}_n(u) du]^{1-\delta'_i}. \quad (6.1)$$

Obviously, by definition of \hat{f}_n , the maximum of (6.1) is $+\infty$ at $h=0$. Hence, the following modified likelihood criterion is considered:

$$\underset{h \geq 0}{\text{maximize}} L_1(h) = \prod_{k=1}^n [\hat{f}_{nk}(z_k)]^{\delta'_k} [\int_{z_k}^{\infty} \hat{f}_{nk}(u) du]^{1-\delta'_k}, \quad (6.2)$$

where

$$\hat{f}_{nk}(z_k) = h^{-1} \sum_{\substack{j=1 \\ j \neq k}}^n s_j K\left(\frac{z_k - z_j}{h}\right).$$

For the standard normal kernel $K(u) = (2\pi)^{-1/2} \exp(-u^2/2)$, the logarithm of (6.2) becomes

$$\begin{aligned} \log L_1(h) = & - \left(\sum_{k=1}^n \delta'_k \right) \log h \\ & + \sum_{k=1}^n \delta'_k \log \left[\sum_{\substack{j=1 \\ j \neq k}}^n s_j (2\pi)^{-1/2} \exp(-(z_k - z_j)^2/2h^2) \right] \\ & + \sum_{k=1}^n (1-\delta'_k) \log \left[\sum_{\substack{j=1 \\ j \neq k}}^n s_j (1-\Phi((z_k - z_j)/h)) \right], \end{aligned} \quad (6.3)$$

where Φ denotes the standard normal distribution function. An approximate (local) maximum of (6.3) with respect to h can be easily found by numerical methods for a given set of censored observations, and this estimated h , denoted by \hat{h}_n , can be used in (3.4) to calculate $\hat{f}_n(x)$.

For this example of the density estimation procedure given by (6.3) and (3.4), the life test data for $n=40$ mechanical switches reported by Nair [38] are used. Two failure modes, A and B, were recorded and Nair estimated the survival function of mode A, assuming the random right-censorship model. Table 1 shows the 40 observations with corresponding δ_1 values, where $\delta_1=1$ indicates failure mode A and $\delta_1=0$ denotes a censored value (or failure mode B). Using this data, the function $\log L_1(h)$ had a maximum in the interval $[0,1]$ at $\hat{h}_{40} \approx 0.18$. Hence, \hat{f}_{40} was computed from (3.4) with bandwidth 0.18. This estimate is shown in Figure 1. This maximum likelihood approach to selecting h_n does not produce the smoothest estimate, but is one criterion that can be used.

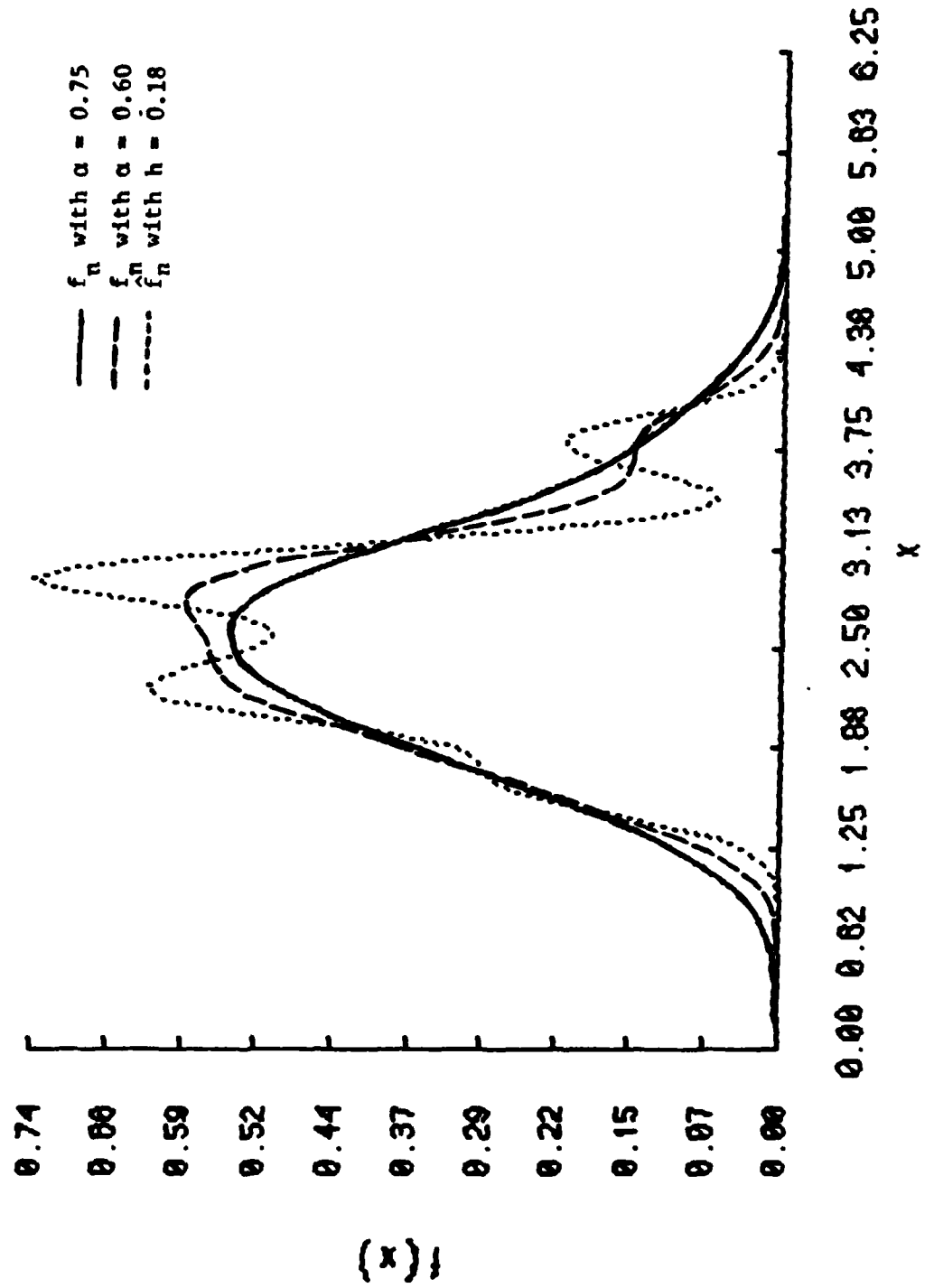
Shown also in Figure 1 are the modified kernel estimates calculated from (3.6) with the " γ_n -nearest neighbor" calculation of Γ_n for the smoothing parameter values $\alpha = 0.60$ and 0.75 . The estimate was also calculated for $\alpha = 0.55$, but was very close to the fixed bandwidth estimate \hat{f}_{40} with $h = 0.18$ and, hence, is not shown. The modified estimator (3.6) with $\alpha = 0.75$ is pleasingly smooth, but with the small sample and only 17 uncensored observations, the value of $\alpha = 0.60$ might be a compromise between the very smooth ($\alpha = .75$) and somewhat rough ($\alpha = .55$) estimates.

TABLE 1. Failure Times (in Millions of Operations) of Switches

z_1	δ'_1	z_1	δ'_1	z_1	δ'_1	z_1	δ'_1
1.151	0	1.667	1	2.119	0	2.547	1
1.170	0	1.695	1	2.135	1	2.548	1
1.248	0	1.710	1	2.197	1	2.738	0
1.331	0	1.955	0	2.199	0	2.794	1
1.381	0	1.965	1	2.227	1	2.883	0
1.499	1	2.012	0	2.250	0	2.883	0
1.508	0	2.051	0	2.254	1	2.910	1
1.543	0	2.076	0	2.261	0	3.015	1
1.577	0	2.109	1	2.349	0	3.017	1
1.584	0	2.116	0	2.369	1	3.793	0

(Fig. 1 here)

Fig. 1 : Density Estimates for Switch Data



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